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MICROSCOPIC AND MACROSCOPIC MODELS IN PLASMA PHYSICS

Harold Grad

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Abstract

In choosing a model to describe the behavior of a plasma, a balance must be maintained between the simplicity of a macroscopic description and the detail in a microscopic description. In an ordinary gas, the criterion for behavior as a continuum is that the mean-free-path be small. In a plasma there is a similar criterion; other lengths (Debye, Larmor) may complicate the macroscopic equations but will not destroy their validity. An entirely different criterion (in a collisionless plasma) is that the Larmor radius be small. A consistent treatment of just the lowest order guiding-center particle motion is sufficient to yield, with a minimum of computation, both a microscopic theory (guiding-center gas) and a macroscopic continuum theory (guiding-center fluid). A comparison shows why certain types of arguments which are conventionally phrased in microscopic terms are exactly equivalent to a potentially less exact macroscopic analysis.

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Errata

p. 7, line 14	For "in the limit of low wavelength" read, "in the limit of large wavelength".
p. 12, eq. 6	For " $\frac{dV}{dt} = \frac{1}{\delta} (\bar{\beta}\cdot\bar{E}) - \mu\bar{\beta}\cdot\nabla\beta$ + etc." read, "$\frac{dV}{dt} = \frac{1}{\delta} (\bar{\beta}\cdot\bar{E}) - \mu\bar{\beta}\cdot\nabla B$ + etc."
p. 16, line 12	For "parallel to U," read, "parallel to $\bar{U}$."
p. 18, lines 11 and 12	For "velocity component U." read, "velocity component $\bar{U}$."
p. 20, line 9	For "Our macroscopic system is now (21)" read, "Our macroscopic system is now (20) and (21)".
p. 26, eq. 33	For "$J = e_+ \int \bar{\xi} f_+ d\bar{\xi} + e_- \int \bar{\xi} f_- d\bar{\xi}$ " read, "$\bar{J} = e_+$ (etc.)"
p. 27, line 11	For "imply $\hat{f}$ only" read, "employ only $\hat{f}$ in this theory".
p. 27, footnote	For "distribution function $\hat{f}(x_1 \dots x_n)$" read, "distribution function $f(x_1 \dots x_n)$".
p. 49, line -2	For "$dt = ds/V = d\bar{\phi}/BV$" read, "$dt = ds/V = d\phi/BV$".

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1. Introduction

One is frequently presented with the alternative of studying a physical problem with a detailed and difficult microscopic model or a less cluttered and simpler, but not so clearly applicable, macroscopic model. The microscopic detail can be not only burdensome but, on occasion, even irrelevant; we recall from statistical mechanics the existence of macroscopic laws that are almost independent of particular molecular properties or motions. For classical fluids, we have recourse to experiment and to the (quite difficult and easily misinterpreted) theoretical derivation of macroscopic laws. For a plasma, unfortunately, experimental knowledge is not yet comprehensive enough to provide any significant degree of synthesis. Thus theoretical derivation of macroscopic laws (at least as difficult and as risky as for an ordinary gas) is our only alternative at present.

The model that one selects to describe a given physical situation is to some extent a matter of choice and may frequently be motivated by one's prejudices or early training. Nevertheless, there do exist objective criteria which should govern the choice between available macroscopic and microscopic descriptions. One may ask why the macroscopic path is ever followed since the microscopic is, in principle, more accurate. Sometimes the justification is that a rough answer is sufficient. At other times, the choice lies between obtaining an answer and none at all; there are many problems which are completely

inaccesible except to macroscopic techniques. Quite generally, the mathematical tools appropriate to a continuum model are more powerful and more highly developed. Finally, there are cases in which the macroscopic analysis becomes exact in an appropriate limit.

In most problems, macroscopic or microscopic, it is necessary to introduce simplifying assumptions. Since the two mathematical frameworks are so different, the appropriate simplifications are usually different in the two formulations, and the results are frequently non-comparable. In problems where the models are comparable, the microscopic solution can be termed "exact", and the relative validity of the macroscopic model determined; such examples will be discussed. However, there are, in the literature, important examples where the simplifications made in the microscopic model are just what is needed to validate an exactly equivalent macroscopic continuum model, even though the comparison may be hidden by a different choice of variables in the two models. In such cases, we shall usually find the macroscopic treatment to be not only easier but more intuitive; investigation of the particle orbits in detail offers only a psychological feeling of greater knowledge. For a guiding-center gas, the comparison of microscopic and macroscopic models is facilitated by proper use of only the lowest order guiding center motion.

Proper choice of a model should, when possible, avoid both the misapplication of an oversimplified continuum treatment and the unnecessary pursuit of the complex paths taken by

individual particles. As two extreme examples in ordinary gas dynamics, we mention the slow steady flow past a sphere and the flow between two rotating cylinders. In the first example, a macroscopic treatment (viz., the Navier-Stokes equations with a slip boundary condition) yields the whimsical result of a finite drag in a vacuum. In the second problem, there does not seem to be the slightest hope of even a qualitative explanation via kinetic theory of the occurrence of instability.

2. Macroscopic Criteria and Regimes

By a macroscopic model we mean one which uses variables similar to those in a thermodynamic or classical fluid description of a state rather than a molecular distribution function. This distinction cannot be made precise since one can bridge the gap with a sequence of more detailed macroscopic descriptions.

In an ordinary gas, the criterion for a macroscopic theory to be valid is that the mean free path be small compared to any natural length over which the gas properties change significantly. Heuristically, molecular collisions will prevent a molecule from straying very far from its neighbors and thus provide a finite fluid element with an identity in terms of the specific molecules it contains. More quantita-

tively, collisions produce a distribution function which is approximately locally Maxwellian and is therefore describable in terms of the local thermodynamic state.

For a plasma, it is necessary to strengthen this criterion somewhat and require that both the mean free path and the mean collision time be short compared to relevant variations in the fluid. In an ordinary gas, slow variation in space implies slow variation in time (with the possible exception of a brief transient period). This distinction can be seen most easily in the dispersion relation for small amplitude plane waves. In an ordinary gas, large wavelength implies low frequency in any propagating mode, but a plasma can also maintain modes (e.g., plasma oscillations) in which there is a finite limiting frequency in the limit of low wavelength.

The situation is further complicated by the existence of several, widely disparate, collision times. In a fully ionized plasma, it is sufficient to have the ion-ion collision time small. This will imply that both the electron and ion distributions are approximately locally Maxwellian although possibly at different speeds and temperatures. Thus the macroscopic description may require separate equations for energy conservation (the ion-electron energy relaxation can be macroscopically slow) but only a single combined momentum equation (the velocity slip or electric current approximates a quasi-equilibrium which is describable by a form of Ohm's law).

The size of a Larmor radius relative to a mean free path

does not affect the possibility of a macroscopic description but only its form. For example, Ohm's law will contain a Hall term unless the electron Larmor frequency is low compared to the mean ion-ion collision time. Similarly, a Debye length which is comparable to the Larmor radius (the Debye length is always small compared to the mean free path) may change the values of the transport coefficients but not the form of the equations. Thus, although there are many forms that the equations can take, and the equations can be quite complex, there are no serious difficulties in obtaining macroscopic equations and no serious doubts as to their range of validity when the mean free path and mean collision time are small.

Appropriate values of transport coefficients can be obtained from either the Boltzmann or the Fokker-Planck equations. For a sufficiently rarefied or hot plasma, the two computations are the same (to "dominant" logarithmic order)[1], whereas for a somewhat denser or cooler plasma (with the logarithmic term no longer dominant), neither can be trusted.

The magneto-ionic theory provides a macroscopic description in an entirely different regime, viz., when there is slow spatial variation but very rapid variation (either transient or oscillatory) in time. In the simplest version, the system consists of Maxwell's equations together with a form of Ohm's law, viz.,

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1. H. Grad, "Theory of Rarefied Gases", in Rarefied Gas Dynamics (Pergamon Press, 1960), para. 23.

(1) $$\tau \frac{\partial \bar{J}}{\partial t} + \bar{J} = \sigma \bar{E} .$$

The fluid appears only as a carrier of current and is represented by the relaxation time τ and conductivity σ as state variables (density and temperature appear in σ and τ and their ratio is the plasma frequency, $\Omega^2 = \sigma/\kappa_o\tau$, κ_o = permittivity of space). This type of macroscopic description can be obtained by approximating moment equations in a low temperature limit[2] or dispersion relations in a high frequency, long wavelength limit; the range of validity is not yet fully understood.

Finally, we mention a third regime in which macroscopic theories are applicable, viz., when fluid variations are slow compared with the Larmor radius and Larmor frequency. The most striking feature of this situation is the strong degree of anisotropy. Particles are rather tightly bound with regard to motion perpendicular to magnetic lines. Thus there is a tendency to maintain the macroscopic identity of a fluid element, but only with regard to motion in the plane transverse to the magnetic field. We shall find this theory to be quite subtle, largely because of the mathematically singular nature of the relevant guiding-center approximation to the exact orbits. As a consequence, even a fully microscopic theory in terms of a distribution function takes on certain macroscopic features.

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2. I. Bernstein, "Linear Wave Phenomena in Collission-Free Plasmas", to appear in Proc. of Polytechnic Inst. of Brooklyn Symp. on Electromagnetics and Fluid Dynamics of Gaseous Plasmas, held in April, 1961.

This is the reason for the otherwise surprising degree of success of the macroscopic theory in certain special cases.

3. Guiding-Center Theory

The motion of a charged particle can be described relatively simply when there is no significant change in the electromagnetic field seen by the particle during one Larmor period, m/eB, and when the energy of the particle does not change appreciably during this same period. This implies a limitation on the rate of time variation of $\bar{E}$ and $\bar{B}$, a limitation on the speed of the particle, and a limitation on the magnitude of $\bar{E}$; (to meet the speed limitation, the component perpendicular to B must not exceed the order of $\mathcal{V}B$, and to meet the energy condition, the parallel component must be small compared to the order of $\mathcal{V}B$, where $\mathcal{V}$ is the maximum allowable speed). The guiding-center motion is obtained as a formal expansion[3] in the small parameter

$$\delta = m/e \; ; \tag{2}$$

one obtains a series which represents the actual motion

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3. a) G. Hellwig, Z. Naturforschung 1, 508 (1955), also

   b) M. D. Kruskal, "The Gyration of a Charged Particle", Project Matterhorn Report PM-S-33, NYO-7903, March 1958.

asymptotically.[4] Consistent with the remarks made above, we must assume that the component of $\bar{E}$ parallel to $\bar{B}$ is small of order δ; (the validity of this condition in an actual plasma will be shown in Section 7).

Entirely different types of behavior are found for the motion of the guiding center and for the rapid oscillation around it; another essential separation occurs between the components of guiding center motion parallel and perpendicular to $\bar{B}$. Briefly, the parallel motion is governed by a second order equation (acceleration given), the perpendicular guiding-center motion by a first order system (velocity given), and the oscillatory motion is completely integrated. To lowest order, the "drift" velocity perpendicular to $\bar{B}$ is given by

$$\bar{U} = \bar{E} \times \bar{B}/B^2. \tag{3}$$

Also to lowest order, the oscillatory motion is described by the statement that the magnetic moment (per mass)

$$\mu = \frac{1}{2} v_{\perp}^2/B \tag{4}$$

is constant; ($v_{\perp}$ is the oscillating perpendicular velocity component left after removing $\bar{U}$). Writing $\bar{v}$ for the total guiding-center velocity and $\bar{V}$ for its parallel component,

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4. J. Berkowitz and C. S. Gardner, Comm. Pure and Appl. Math. 12, 501 (1959).

(5) $$\bar{v} = \dot{\bar{U}} + \bar{V}\ , \quad \bar{V} \times \bar{B} = 0,$$

we complete the lowest order description with the equation for the parallel motion

(6)
$$\frac{dV}{dt} = \frac{1}{\delta}\,(\bar{\beta}\cdot\bar{E}) - \mu\bar{\beta}\cdot\nabla\beta + \bar{U}\cdot\left(\frac{\partial\bar{\beta}}{\partial t} + \bar{U}\cdot\nabla\bar{\beta} + \bar{V}\cdot\nabla\bar{\beta}\right)$$
$$= \frac{1}{\delta}\,(\bar{\beta}\cdot\bar{E}) + \bar{\beta}\cdot\nabla\left(\frac{1}{2}\,U^2 - \mu B\right) + V(\bar{U}\cdot\bar{\kappa}).$$

Here $\bar{\beta}$ is the unit tangent vector and $\bar{\kappa}$ the curvature of the magnetic line,

(7)
$$\begin{cases} \bar{\beta} = \bar{B}/B \\ \bar{\kappa} = (\bar{\beta}\cdot\nabla)\bar{\beta}. \end{cases}$$

Note that the parallel component of $\bar{E}$ exerts a finite effect on the motion even though it is a higher order quantity.

Since $\bar{\beta}\cdot\bar{E}$ is small, $\bar{U}$ satisfies the relation

(8) $$\bar{E} + \bar{U} \times \bar{B} = 0$$

to lowest order. From Maxwell's equation, we get the familiar flux equation,

(9) $$\frac{\partial\bar{B}}{\partial t} + \text{curl }(\bar{B} \times \bar{U}) = 0.$$

This equation has the well-known property of carrying magnetic

lines into magnetic lines. It is therefore possible to assign an identity to a magnetic line and follow its motion in time. Furthermore, one can interpret a guiding center as being constrained to move along a specified magnetic line which is itself in motion at the velocity $\overline{U}$; (this has sometimes been questioned[5]). Introducing the line as a constraint with σ as a parameter along the line (σ is constant for a point moving with velocity $\overline{U}$), we find the motion to be describable by the Lagrangian and Hamiltonian

$$(10)\qquad \begin{cases} L = \frac{1}{2}\,\dot{\sigma}^2/\zeta^2 + \frac{1}{2}\,U^2 - \mu B - \phi^*/\delta, \\[2ex] H = \frac{1}{2}\,\zeta^2 p^2 - \frac{1}{2}\,U^2 + \mu B + \phi^*/\delta. \\[2ex] \zeta = \frac{\partial\sigma}{\partial s}\,, \quad \dot{\sigma} = \zeta V = \zeta^2 p\ . \end{cases}$$

Here ϕ^* is the potential of the parallel component of $\overline{E}$.[6]

The lowest order theory described above is by itself sufficient to allow the derivation of a complete macroscopic or microscopic theory of a guiding-center plasma, but, for purposes of later discussion, we shall find it illuminating to list the perpendicular "drift" velocity to the next order,

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5. M. N. Rosenbluth and C. L. Longmire, Ann. of Phys. 1, 120 (1957).

6. This potential may be defined in an arbitrary time-varying field, e.g., by taking $B = \nabla\alpha \times \nabla\gamma$ = curl A and $A = \alpha\,\nabla\gamma$ from which $E = -\partial A/\partial t - \nabla\phi = E^* - \nabla\phi^*$ where $E^* = (\partial\gamma/\partial t)\,\nabla\alpha - (\partial\alpha/\partial t)\,\nabla\gamma$ $(E^*\cdot B = 0)$ and $\phi^* = \phi + \alpha\ \partial\gamma/\partial t$.

viz.[7],

$$\bar{U}_1 = \bar{U} + \frac{\delta}{B^2} \bar{B} \times [\frac{d\bar{v}}{dt} + \mu \nabla B]$$

(11)

$$= \bar{U} + \frac{\delta}{B^2} [\frac{d\bar{E}}{dt} + v \times \frac{d\bar{B}}{dt} + \mu\bar{B} \times \nabla B];$$

here the time derivative is taken following the lowest order guiding-center motion,

(12) $$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{v}\cdot\nabla = \frac{\partial}{\partial t} + \bar{V}\cdot\nabla + \bar{U}\cdot\nabla \ .$$

We shall see in examples below that the first order drifts add very little to the information that is already contained in the lowest order description.

4. Examples

Consider a unidirectional steady magnetic field in the z-direction with its magnitude B a function of x. There is a first-order drift velocity in the y-direction of magnitude

(13) $$v_d = \delta\mu \frac{d}{dx} (\log B).$$

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7. See H. Grad, "A Guiding Center Fluid", AEC Report TID-7503, Feb. 1956, p. 495. Many published expressions for this drift (and for the parallel motion (6)) are either incomplete or incorrect.

If there is a uniform electric field in the y-direction, there is an additional zero-order drift in the x-direction of magnitude $U = E/B$. The energy of a particle is not constant in the presence of an electric field. From the lowest order guiding-center motion, only the "internal" energy μB changes in virtue of the drift $\overline{U}$ which carries the particle to a region of different B. From the exact (not guiding-center) equations of motion, one would compute the energy change as $e\overline{E}\cdot\overline{v}$; inserting the guiding-center approximation into this formula yields an energy change $e\overline{E}\cdot\overline{v}_d$. The two energy computations, of course, yield the same result,

$$U \frac{d}{dx} (m\mu B) = ev_d E = \frac{1}{2} mv_{\perp}^2 \, E \frac{d}{dx} (\log B). \tag{14}$$

But the interpretations are worlds apart. The lowest order guiding-center theory, which expressly ignores the existence of the (higher order) drift in the direction of $\overline{E}$, yields the correct energy balance as a change in "internal" energy due to the lowest order drift into a region of different B. One must not supplement this with an <u>additional</u> energy increment due to the first order drift into a region of different electric potential. This examination of higher order terms only gives insight into the mechanism by which the zero-order guiding-center theory keeps its bookkeeping straight.

We have tacitly assumed above (just for simplicity) that U is small (neglecting the guiding-center energy component $\frac{1}{2} U^2$). The complete argument involves a guiding-center energy

change in $\mu B + \frac{1}{2} U^2$ and the complete drift from (11) which has a term involving $d\bar{U}/dt = (\bar{U}\cdot\nabla)\bar{U}$ as well as a gradient of B.

We can invert the argument given above and compute the correct first-order drift velocity using only the zero-order theory and the conservation of energy. One can even use this argument in complete generality, using the zero-order motion (6) together with the drift $\bar{E} \times \bar{B}/B^2$ and constancy of the magnetic moment, to obtain the first-order drift (11) as a consequence of energy conservation; (the computation is intricate, but not difficult). There is one gap, viz., an undetermined drift of <u>first order</u> parallel to U. It is possible that a systematic treatment of higher order energy balances would eliminate this gap; but to the order computed, we can only say that the first order drifts are compatible with energy conservation and are almost a direct consequence.

As another example, take the field surrounding a straight wire carrying a constant current. Since $B \sim 1/r$, there is a field gradient which produces the axial first-order guiding-center drift

$$v_d' = \delta\mu/r = \frac{1}{2} v_\perp^2(\delta/rB). \tag{15}$$

But there is also a guiding-center acceleration of magnitude $|d\bar{v}/dt| = v^2/r$ due to the circular motion, and this causes an additional axial drift of magnitude (cf. (11)),

(16) $$v_d'' = V^2(\delta/rB);$$

the two drifts are additive,

(17) $$v_d = (V^2 + \tfrac{1}{2}\, v_\perp^2)(\delta/rB).$$

Now let us apply a constant small axial electric field. This superposes a radial drift U, but it also modifies the parallel motion according to (6),

(18) $$\frac{dV}{dt} = V(\overline{U}\cdot\overline{\kappa}) = V\, U/r \;;$$

(this expresses the conservation of angular momentum for a particle constrained to move on a circle of variable radius). The lowest order guiding-center energy of the particle is $\frac{1}{2}\, V^2 + \mu B$ (neglecting $\frac{1}{2}\, U^2$). Exactly as in the first example, the change in μB resulting from the radial U drift is equal to the change in electric potential produced by the axial "gradient" drift v_d'. Similarly, the change in $\frac{1}{2}\, V^2$ computed from (18) is equal to the change in electric potential produced by the axial drift v_d''.

The principal conclusion to be drawn from these examples is that the lowest order guiding-center motion is completely self-contained, even though the mechanism by which this is accomplished may be unfamiliar and is hidden in first-order terms which are expressly excluded. There is also a caution that one

should not misapply to an approximate theory familiar conceptions which are appropriate only to the exact equations of motion.

5. The Guiding-Center Fluid.[8]

We consider a plasma in which it is believed that most of the particles follow orbits which can be described by the guiding-center theory. We shall give a very simple heuristic derivation of a macroscopic fluid theory using only the most elementary properties of the motion of the individual particles.

We have seen that every particle, independent of its mass, charge, or energy, has the same perpendicular velocity component U. Thus the fluid velocity, $\bar{u}$, no matter how it may be defined, can only differ from $\bar{U}$ by a component parallel to $\bar{B}$, and the fluid is therefore perfectly conducting,

$$\bar{E} + \bar{u} \times \bar{B} = 0, \tag{19}$$

and flux preserving,

$$\frac{\partial \bar{B}}{\partial t} + \text{curl } (\bar{B} \times \bar{u}) = 0. \tag{20}$$

8\. a. H. Grad, "A Guiding Center Fluid", AEC Report TID-7503, Feb. 1956, p. 495; see also

b. M. Goldberger, "One Fluid Hydromagnetics and the Boltzmann Equation", ibid, p. 260, and

c. G. Chew, M. Goldberger and F. Low, Proc. Roy. Soc. (London) A236, 112 (1956).

We now write the macroscopic laws of conservation of mass, momentum (with the usual magnetic Lorentz term as befits a good conductor), and internal energy (heuristically ignoring the heat flow),

$$
(21) \qquad \begin{cases} \dfrac{d\rho}{dt} + \rho \operatorname{div} \bar{u} = 0 \\[2ex] \rho \dfrac{d\bar{u}}{dt} + \operatorname{div} \mathbb{P} = \bar{J} \times \bar{B} \\[2ex] \rho \dfrac{d\varepsilon}{dt} + P_{ij} \dfrac{\partial u_i}{\partial x_j} = 0 \ , \end{cases}
$$

$$
(22) \qquad \frac{d}{dt} = \frac{\partial}{\partial t} = \bar{u} \cdot \nabla \ .
$$

Here $\mathbb{P}$ or P_{ij} is the full stress tensor, and ε is the internal energy.

Now, the rapid spiraling about a guiding center implies isotropy of the stress tensor in the plane perpendicular to $\bar{B}$; in a coordinate system with axes parallel and perpendicular to $\bar{B}$,

$$
(23) \qquad \mathbb{P} = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_2 \end{pmatrix}
$$

Expressed invariantly,

$$
(24) \qquad P_{ij} = p_2(\delta_{ij} - \beta_i\beta_j) + p_1\beta_i\beta_j \ .
$$

We may define temperatures by

$$(25)\qquad \begin{cases} p_1 = \rho R T_1 \\ \\ p_2 = \rho R T_2 \ . \end{cases}$$

The internal energy per unit mass ε (which would be $\frac{3}{2}$ RT in a conventional gas) is evidently given by

$$(26)\qquad \varepsilon = \frac{1}{2} RT_1 + RT_2 = p_1/2\rho + p_2/\rho$$

since there is one parallel degree-of-freedom and two perpendicular.

Our macroscopic system is now (21) with $\mathbb{P}$ and ε replaced in terms of p_1, p_2, and $\bar{\beta}$. Setting $J = (1/\mu_o)\text{curl } \bar{B}$, we have a system of equations in the variables, ρ, p_1, p_2, $\bar{u}$, $\bar{B}$ which seems to be very similar in appearance to those of conventional magneto-fluid dynamics, except for the occurrence of two pressures. To complete the system, one more equation is needed. We use the microscopic fact that the magnetic moment of an individual particle is constant. Macroscopically, we state that the rate of change of total magnetic moment in a fluid element is given by the net rate of flow of magnetic moments carried through the surface. The mean magnetic moment (per mass) is

$$(27)\qquad \langle\mu\rangle = RT_2/B = p_2/\rho B \ .$$

Thus we have

$$\frac{d}{dt}\,(p_2/\rho B) = 0, \tag{28}$$

on neglecting the "heat" flow of magnetic moment relative to the mean flow, $\bar{u}$.

By counting, the system of equations is now complete. An easy manipulation shows (just as in ordinary gas dynamics) that the energy equation can be replaced by an entropy equation, viz.,

$$\frac{d}{dt}\,(p_1 p_2^2/\rho^5) = 0. \tag{29}$$

This is to be compared with p^3/ρ^5 which is ordinarily constant on a particle path. Our system of equations, finally, consists of the mass, momentum, and flux equations, together with two particle path invariants which can be chosen for convenience as any two of the following

$$p_2/\rho B\ ,\quad p_1 B^2/\rho^3\ ,\quad p_1 p_2^2/\rho^5\ ,\quad p_2^3/p_1 B^5\ . \tag{30}$$

One can discuss the validity of this system of equations in two distinct contexts: one is by comparison with a presumably better theory; the other is an intrinsic question of the mathematical suitability of the system, *per se*. The second is more urgent, since it is perfectly possible to arrive at a mathematically incompatible or otherwise unsuitable

system by plausible physical reasoning. To be precise, the equations themselves can specify situations in which they are definitely invalid. This comes about through the fact that, unless certain inequalities are satisfied, the initial value problem is not well posed. The forbidden regions (non-overlapping) are given by[9]

$$(31) \quad \begin{aligned} &(a) \quad p_2 + B^2/\mu_o < p_1 \\ &(b) \quad 3p_1 \, B^2/\mu_o < p_2(p_2 - 6p_1). \end{aligned}$$

Roughly speaking, one cannot have p_1 too large (a) or p_2 too large (b). These relations were originally found by a linearized plane wave analysis and interpreted as conditions for instability.[10] Although this is not a logical deduction, it may be heuristically sound; viz., a complete breakdown of otherwise plausible equations may be indicative of violent behavior in fact. It is worth remarking that the inequalities (31) describe a breakdown of the equations in complete generality; the reversed inequalities must be satisfied at every point of a general, nonsteady, nonlinear flow.

Provided that the inequalities (31) are avoided, this system of equations is similar in mathematical structure to

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9. This is obtained by computing the characteristics of the system of differential equations.

10. R. Lüst, "On the Stability of a Homogeneous Plasma with Non-isotropic Pressure", AEC Report TID-7582, Nov. 1959.

the conventional perfectly conducting isotropic fluid system. It should be observed that the constraint $p_1 = p_2$ does not reduce the guiding-center system to the conventional one. Although the momentum and energy equations are the same, there is an extra particle path invariant, viz., $p_2^3/p_1B^5 \sim p^2/B^5$, and this will not be invariant, in general, for a solution of the conventional equations. In other words, a guiding-center problem which is provided with boundary conditions and initial conditions which are compatible with isotropy of the stress tensor does not necessarily have an isotropic solution.

There is a special type of flow which does reduce to conventional magneto-fluid flow precisely, viz., a transverse flow in which the magnetic field is unidirectional and the flow lies in the plane perpendicular to $\bar{B}$. The pressure p_2 is isotropic in the relevant plane, and p_1 does not enter. Furthermore, there is an exact mathematical analogue in this geometry between ordinary compressible fluid flow (non-magnetic) and magneto-fluid flow;[11] this analogue therefore extends to the guiding-center flow. Finally, since the mechanism which we expect to validate any macroscopic theory, viz., the coherence of the molecules within a given fluid element, is satisfied in this geometry, we expect the macroscopic theory to closely approximate the exact microscopic theory, provided only that the conditions for the validity of the guiding-center approximation are satisfied. This will be

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11. H. Grad, Revs. Mod. Phys. 32, 830 (1960).

verified.

The conditions for the validity of the guiding-center fluid equations in general three-dimensional problems are quite subtle. For example, one might think that in an <u>approximately</u> transverse flow with only slow variation along the magnetic field direction, the equations would be approximately valid.[12] This is not so; the reason is the non-local behavior of the particle motion along the field lines. The velocity distribution at a certain point may be determined by the maximum value of B along the line (mirror ratio), even though this maximum may be attained at a great distance with correspondingly small field gradients. A simple example in which the macroscopic theory can be compared with the "exact" particle theory is in a nozzle with slowly varying area. One finds the speed at the throat to be $(3RT_1)^{1/2}$ for the fluid and $(\frac{2}{\pi} RT_o)^{1/2}$ for the particle model; (T_o is the entering temperature).

6. <u>The Guiding-Center Gas</u>

We again consider a gas in which most of the particle orbits satisfy the conditions for the validity of the guiding-center theory. But instead of a macroscopic theory, we now wish to derive a microscopic theory in terms of an appropriate

12. For example, see footnote Ref. 8c.

molecular distribution function. Our attitude will be that the lowest-order guiding-center approximation defines the exact motion of an artificial guiding-center particle. This gas can be described microscopically in terms of a guiding-center distribution function. It is this interpretation, viz., that one constructs an exact equation for a distribution function which represents a gas of artificial particles[13], which offers the simplification of allowing the use of the lowest order guiding-center theory alone, as distinguished from roughly equivalent theories in which a similar expansion in the parameter δ is introduced into the exact Liouville equation in which the orders of expansion in δ are inextricably mixed.[14]

The most direct interpretation is to consider a guiding-center particle as a molecule located at its guiding center with the oscillation about the guiding center representing internal energy, magnetic moment, and angular momentum of the molecule. With this interpretation, we are dealing with a magnetic material, also with a one-dimensional gas which has thermal motion only along the field lines, and a pressure tensor

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13. H. Grad, footnote Ref. 8a.

14. a) K. M. Watson, Phys. Rev. 102, 12 (1956).

    b) K. A. Brueckner and K. M. Watson, Phys. Rev. 102, 19 (1956).

    c) S. Chandrasekhar, A. N. Kaufman, and K. M. Watson, Ann. of Physics 2, 435 (1957).

    d) M. N. Rosenbluth and N. Rostoker, Phys. Fluids 2, 23 (1959).

with only a single non-vanishing component, $P_{ij} = p_1\beta_i\beta_j$. All macroscopic consequences will be equivalent to those in the more conventional interpretation as a three-dimensional gas, but a dictionary of equivalents is required, e.g., to translate internal energy into p_2, etc. Although we have a choice of interpretations macroscopically, the only natural microscopic choice is as the less familiar one-dimensional gas because of the very singular nature of the laws of motion of a guiding-center particle.

The conventional microscopic treatment of a fully ionized collisionless plasma is to introduce a distribution function $f(\bar{\xi}, \bar{x}, t)$ (normalized as a number density) which satisfies the Liouville equation

$$\frac{\partial f}{\partial t} + \bar{\xi} \cdot \frac{\partial f}{\partial \bar{x}} + \frac{e}{m}(\bar{E} + \bar{\xi} \times \bar{B}) \cdot \frac{\partial f}{\partial \bar{\xi}} = 0. \tag{32}$$

There are two such equations for $f_\pm$ with parameters $m_\pm$ and $e_\pm$ (signed). To complete the system, we take Maxwell's equations where the source terms are given by

$$\begin{aligned} J &= e_+ \int \bar{\xi}\, f_+ d\bar{\xi} + e_- \int \bar{\xi}\, f_- d\bar{\xi} \\ q &= e_+ \int f_+ d\bar{\xi} + e_- \int f_- d\bar{\xi}\,. \end{aligned} \tag{33}$$

The equations of conservation of mass, momentum, and energy (with full stress tensor and heat flow) are rigorous consequences of this system.

To describe a guiding-center gas, we introduce a distribution function $\hat{f}(V, \mu, \bar{x}, t)$ as a number density. We compute the Jacobian of the transformation from $(\bar{\xi})$ to (V,μ),

$$d\bar{\xi} = d\xi_1 d\xi_2 d\xi_3 = dV\, dv_\perp^1\, dv_\perp^2 = \pi dV d(v_\perp^2) = 2\pi B\, dV\, d\mu, \tag{34}$$

and from the approximate identification $f d\bar{\xi} \sim \hat{f}\, dV\, d\mu$, we conclude that

$$\hat{f} \sim 2\pi B f. \tag{35}$$

This lowest order identification can be used to compute density, also pressure (with care), but current not at all (this will be seen). We shall make no use of this connection and shall imply $\hat{f}$ only.

Taking the lowest-order guiding-center motion, we write the Liouville equation for $\hat{f}$,[15]

$$\frac{\partial\hat{f}}{\partial t} + \frac{\partial}{\partial\bar{x}} \cdot [(\bar{U} + \bar{V})\hat{f}] + \frac{\partial}{\partial V}\Big\{[\frac{1}{\delta}(\bar{\beta}\cdot\bar{E}) + \bar{\beta}\cdot\nabla(\frac{1}{2}U^2 - \mu B) + V(\bar{U}\cdot\bar{\kappa})]\hat{f}\Big\} = 0. \tag{36}$$

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15. For a distribution function $\hat{f}(x_1 \ldots x_n)$ in an arbitrary phase space in which the equations of motion are $dx_r/dt = X_r$, the Liouville equation is $\partial f/\partial t + \sum \frac{\partial}{\partial x_r}(X_r f) = 0$. If $\operatorname{div} X = 0$ (e.g., in a Hamiltonian system), then $\frac{\partial f}{\partial t} + \sum X_r \frac{\partial f}{\partial x_r} = 0$, which states that f is constant on a particle path.

Note that in these variables $\hat{f}$ is not constant on a particle path. The independent variables should be kept in mind; e.g., $\bar{U}$ is a function of $\bar{x}$ and t, $\bar{V} = V\beta$ is a function of V and of x and t (through β), etc.

It is also possible to introduce intrinsic coordinates attached to a moving magnetic line. We can write

$$(37) \qquad \bar{B} = \nabla\alpha \times \nabla\gamma$$

where α and γ are constant on a given moving magnetic line, and we also introduce a parameter σ along each line in such a way that σ is constant at a point moving with velocity $\bar{U}$; (α, γ, and σ are fixed Lagrangian coordinates relative to the motion $\bar{U}$). The Jacobian of the transformation from ($\bar{x}$) to (α, γ, σ) is obtained from $\bar{B}\cdot d\bar{S} = d\alpha\, d\gamma$ = flux, and (10),

$$(38) \qquad d\bar{x} = dx_1 dx_2 dx_3 = \frac{1}{B}\, d\alpha\, d\gamma\, ds = \frac{1}{B\zeta}\, d\alpha\, d\gamma\, d\sigma.$$

Introducing the number density $\tilde{f}(p, \mu, \alpha, \gamma, \sigma, t)$, where p is the momentum conjugate to σ, (10), we have

$$(39) \qquad \tilde{f} \sim \hat{f}/B \sim 2\pi f$$

and the Liouville equation

$$(40) \qquad \frac{\partial\tilde{f}}{\partial t} + p\zeta^2 \frac{\partial\tilde{f}}{\partial\sigma} + \frac{\partial}{\partial\sigma}\left[\frac{1}{2}U^2 - \mu B - \phi^*/\delta - \frac{1}{2}p^2\zeta^2\right]\frac{\partial\tilde{f}}{\partial p} = 0.$$

It is tempting to replace σ by the arc length s and set $\zeta = 1$, $p = V$. This can be done at any given instant, but we must remember that $\partial\tilde{f}/\partial t$ is the rate of change of $\tilde{f}$ with σ fixed which is not the same as with s fixed if, e.g., the magnetic line is stretching.

The most striking feature of all of these guiding-center Liouville equations is that they are incomplete. From the distribution function one can compute a parallel component of velocity $\langle V \rangle = \bar{\beta} \cdot \bar{u}$, and from the Liouville equation an equation for the parallel component of momentum, i.e., an equation for $d\langle V \rangle/dt$. But the temporal evolution of the perpendicular component, $\bar{U}$, is not obtainable. The Liouville equation only describes the motion of particles which are constrained to lie on a magnetic line, assuming that the motion of the line is given. The system must be supplemented by an equation describing $d\bar{U}/dt$.

Let us tentatively introduce as the equation determining the transverse motion, the perpendicular component of the macroscopic momentum equation (21), viz.,

$$(41) \qquad \rho\bar{B} \times \frac{d\bar{u}}{dt} + \bar{B} \times \operatorname{div} \mathbb{P} = \frac{1}{\mu_o} \bar{B} \times \operatorname{curl} \bar{B} \times \bar{B}.$$

It is easy to see that this does specify $d\bar{U}/dt$. The omitted component, $\bar{B} \cdot d\bar{U}/dt$ is equal to $-\bar{U} \cdot d\bar{B}/dt$, and $d\bar{B}/dt$ is already given by the flux equation. Also, the term $\bar{B} \times d\langle \bar{V} \rangle/dt$ which is contained in (41) is obtainable from the Liouville equation.

The conventional description of a plasma consists of equations describing the particle motion (32), Maxwell's equations, and the connecting source relations (33). In our case, the source relations are very singular. In passing from macroscopic mass flow, $\rho\bar{u} = mn\bar{u}$, to macroscopic current flow, $\bar{J} = en\bar{u}$, it is necessary to multiply by the large quantity $e/m = 1/\delta$, thereby changing the order of the term. The parallel component of current, $\bar{\beta}\cdot\bar{J}$, which can be computed directly as a difference of moments of the two distribution functions, would seem to be very large, of order $1/\delta$, thereby nullifying the entire theory. To avoid this disaster, we must impose the constraint that $\langle V\rangle_+ = \langle V\rangle_-$ and hope that this can be done compatibly within the framework of our theory. Fortunately there is the small electric field component, $\bar{\beta}\cdot\bar{E}$, which should not appear in the theory but does. Tentatively, we can try to adjust the value of $\bar{\beta}\cdot\bar{E}$ so as to validate the constraint. Specifically we take the two moment equations for $d\langle V\rangle_+/dt$ and $d\langle V\rangle_-/dt$ and set them equal. We obtain the result (familiar as a form of Ohm's law)

$$(1/\delta_+ + 1/\delta_-)(\bar{\beta}\cdot\bar{E}) = \bar{\beta}\cdot\mathrm{div}\,\mathbb{P}_+/\rho_+ - \bar{\beta}\cdot\mathrm{div}\,\mathbb{P}_-/\rho_- . \tag{42}$$

In principle, this expression for $\bar{\beta}\cdot\bar{E}$ should be used in computing individual orbits, (6), as well as in a plasma theory.

At this stage, the current component $\bar{\beta}\cdot\bar{J}$ has completely disappeared; it is zero to order $1/\delta$ and it is not computable

to finite order within the theory. A glance shows that the perpendicular component of current has the same status. To order $1/\delta$, there is no current since the drift $\bar{U}$ is the same for ions and electrons; (the result given above, $\langle V \rangle_+ = \langle V \rangle_-$, also implies that $\bar{u}_+ = \bar{u}_-$, which is compatible with charge neutrality to lowest order). While it is true that a zero-order drift current, $\bar{J}_d$, can be computed from the first order drifts (11), this is technically not contained within our theory. To be complete, we would also add the polarization current, $\bar{J}_m = \text{curl } \bar{M}$, which is produced by the mean magnetic moment per volume,

$$(43) \qquad \bar{M} = -\bar{\beta} \left\langle \frac{1}{2} \, mnv_{\perp}^{2}/B \right\rangle = -\bar{\beta} \; p_2/B = -\bar{B} \; p_2/B^2 \; .$$

Fortunately, it is not at all necessary to compute currents from the particle motions; we can simply use the relation $\bar{J} = (\text{curl } \bar{B})/\mu_o$; (this has already been introduced in (41)). In summary, a compatible determinate system describing the lowest-order guiding-center gas is given by a pair of Liouville equations (36) in which the value (42) for $\bar{\beta}\cdot\bar{E}$ is inserted, together with the constraint equation (41) for the perpendicular motion and the flux equation (20). The higher order drifts and electric fields which we ignore will presumably adjust themselves appropriately to yield the proper lowest order constraints, (41) and (42); this will be examined in the next section.

A procedure which appears to be more accurate than the one just described is to take a full Lorentz force term, $q\overline{E} + \overline{J} \times \overline{B}$, and include the displacement current. The net effect is to include additional Maxwell stress and electromagnetic momentum terms in (41), $\overline{E}$ being replaced by $\overline{B} \times \overline{u}$. The additional terms have the order A^2/c^2 (A is the Alfvèn speed, see (58)). Our tacit assumption has been that the plasma density is high enough so that this factor is small; (this parameter is entirely distinct from the small parameter δ). Inclusion of these extra terms may possibly improve the results for very low density plasmas, but the situation is not clear. Whereas the charge-neutral formulation is consistently Galilean-invariant, the alternative system has no invariance properties at all and shows rather awkward behavior in ordinary magneto-fluid dynamics.[16] The only proper way to cover the entire range of densities is by a completely relativistic treatment, but this has not yet been done.

The guiding-center treatment of the plasma is seen to be essentially macroscopic with regard to the motion of the magnetic field lines, but it is microscopic with regard to the particle motion along the lines. This is consistent with our original heuristic argument regarding the preservation of the identity of a fluid element. The value chosen for $\overline{\beta} \cdot \overline{E}$, viz., (42), is such as to guarantee $\overline{u}_+ = \overline{u}_-$ provided that

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16. A. A. Blank and H. Grad, "Fluid Magnetic Equations", Institute of Mathematical Sciences Report, NYO-6846-VI, July 1958.

this holds initially; in other words there is a single macroscopic fluid velocity. From the constraint equation (41) and the moment equation for $d\langle V \rangle/dt$, one obtains exactly the same momentum equation as in the fluid theory (21). The mass equation is also the same, but the energy equation differs in the pressence of a heat flow component which is parallel to $\bar{B}$. Apparently, the only advantage of the microscopic theory is that it more correctly takes into account heat flow and magnetic moment flow. The "only" is misleading, however; the superiority of the classical Boltzmann equation over fluid dynamics also lies only in the treatment of the stresses and heat flow.

It is a simple matter to verify that the behavior is entirely macroscopic in a transverse flow, i.e., one in which the magnetic field is unidirectional, and the spatial variation and flow are perpendicular to the field; one only needs to observe that there is neither heat flow nor magnetic moment flow in the transverse plane. In this geometry, one can obtain higher order guiding-center equations (as an expansion in δ)[17] just as one obtains the Navier-Stokes and other equations as an expansion in mean free path in the Chapman-Enskog theory of an ordinary gas.

17. Marion H. Rose, "On Plasma Magnetic Shocks", Institute of Mathematical Sciences Report NYO-2885, March 1960.

7. Reconciliation

The lowest order guiding-center theory in terms of the distribution function $\hat{f}$ is quite dissimilar to the exact theory in terms of f, especially with regard to the coupling with Maxwell's equations. The simplicity of the preceeding analysis lay in choosing constraint equations for $\beta\cdot\bar{E}$ and $d\bar{U}/dt$ instead of current source relations to complete the system. It is not difficult to verify that if one computes drift currents using (11), adds the polarization current, and sums over both fluids, one obtains exactly the constraint equation (41). But a simpler procedure is to work with the exact momentum equations for the ions and electrons individually as obtained from the full (not guiding-center) Liouville equation (32), viz.,

$$(44) \qquad \frac{d\bar{\bar{u}}}{dt} + \frac{1}{\rho}\,\mathrm{div}\,\bar{\mathbb{P}} = \frac{1}{\delta}\,(\bar{E} + \bar{\bar{u}} \times \bar{B}).$$

We distinguish the exact velocity $\bar{\bar{u}}$ from the zero-order guiding-center velocity $\bar{u}$; indices $(\pm)$ are omitted. Consistent with the guiding-center ordering in δ, we obtain $\bar{E} + \bar{u} \times \bar{B} = 0$ to lowest order, and obtain the next order on the right side of (44) by substituting the lowest order expressions on the left. In terms of

$$(45) \qquad \bar{J} = \rho(\bar{\bar{u}}-\bar{u})/\delta$$

(correct for the perpendicular component only), we get

$$(46) \qquad \bar{B} \times [\rho \frac{d\bar{u}}{dt} + \text{div } \mathbb{P}] = \bar{B} \times \bar{J} \times \bar{B}.$$

We have obtained approximately the same equation for ions and electrons individually that was previously postulated for the entire fluid (41); the crucial distinction is that $\bar{J}$ in (46) is $\bar{J}_+$ or $\bar{J}_-$ and cannot be set equal to curl $\bar{B}/\mu_o$. But we can reverse the argument and solve (46) for $\bar{J}_+$ and $\bar{J}_-$ in terms of zero order guiding center quantities; subtracting curl $\bar{M}$, we would find exactly the same expression for $\bar{J}_d$ as that obtained from the drift calculation. The constraint equation (41) is obtained from the sum of the two equations (46), replacing the total $\bar{J}$ by (curl $\bar{B})/\mu_o$.

In other words, computation of J from the first order drifts (not in the theory) and curl $\bar{M}$ is exactly compatible with the constraint equation (41) which was postulated. If the individual currents $\bar{J}_d^+$ and $\bar{J}_d^-$ are desired for any purpose, they can be obtained from the individual momentum equations which, otherwise, contribute nothing new.

Now we examine the parallel component of the exact momentum equation (44). Proceding as before, we would find $\bar{\beta}\cdot\bar{E} = 0$ to lowest order and evaluate $\bar{\beta}\cdot\bar{E}$ to first order in terms of a zero order left side of the equation. But, if we choose not to impose the constraint that $\bar{\beta}\cdot\bar{u}_+ = \bar{\beta}\cdot\bar{u}_-$ to lowest order (temporarily violating the expansion in δ) we find (approximating slightly)

$$(47) \qquad \bar{\beta}\cdot \frac{\partial \bar{J}}{\partial t} + \frac{1}{\delta_-}\bar{\beta}\cdot\text{div } \mathbb{P}_- = \Omega^2 \kappa_o \bar{\beta}\cdot\bar{E} .$$

This should be compared with (42) and also with

$$-\kappa_o \frac{\partial \overline{E}}{\partial t} + \frac{1}{\mu_o} \operatorname{curl} \overline{B} = \overline{J} . \tag{48}$$

The two equations (47) and (48) describe plasma oscillations (presumably very fast) about an equilibrium given by (42) for $\overline{\beta} \cdot \overline{E}$ and curl $\overline{B}/\mu_o$ for $\overline{J}$. A large amplitude plasma oscillation would interfere with our basic requirements for guiding-center motion. But, for slowly varying "source" terms div $\mathbb{P}$ and curl $\overline{B}$, the generated plasma oscillations will be small in amplitude, and the quasi-equilibrium for both $\overline{\beta} \cdot \overline{E}$ and $\overline{\beta} \cdot \overline{J}$ will be set up after a short transient comparable to a plasma period.[18]

The situation is more complicated if one includes displacement current and considers problems in which the plasma frequency is not high compared to the Larmor frequency. In such a case one must impose as an additional requirement for the validity of the guiding-center theory that all natural motions be slow compared to the plasma frequency as well as the Larmor frequency. Otherwise, variations in the "source" terms div $\mathbb{P}$ and curl $\overline{B}$, will produce inappropriately large values of $\overline{\beta} \cdot \overline{J}$ and $\overline{\beta} \cdot \overline{E}$. Thus plasma oscillations must be expressly excluded from any guiding-center theory.[19]

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18. H. Grad, "Ohm's Law", Institute of Mathematical Sciences Report NYO-6486-IV, August 1956.

19. The parallel component of E is frequently mishandled; both by omission, (e.g., see footnote Refs. 7 and 14b) and by excess (including plasma oscillations, e.g., see footnote Ref. 14c).

8. Examples

In this section we give a number of representative examples to show the complex interplay between microscopic and macroscopic interpretations and, specifically, the confusion that can arise in the ordering with respect to δ.

To start, we generalize the first example of Section 4 and consider a unidirectional z-field, $\bar{B}$, which is a function of both x and y. The drift velocity (11) produces a macroscopic current

$$\bar{J}_d = \frac{p_2}{B^2} \bar{\beta} \times \nabla B. \tag{49}$$

To this we must add a current $J_m = \text{curl } M$ from (43), viz.,

$$\bar{J}_m = \bar{\beta} \times \nabla (p_2/B). \tag{50}$$

There is a "drift" component in the direction of the contour of constant B and a "polarization" component along the contour of constant p_2/B. The total current, $\bar{J} = \bar{J}_m + \bar{J}_d$ is easily seen to be in the direction of the contour p_2 = constant; quantitatively,

$$\bar{J} = \frac{1}{B} \bar{\beta} \times \nabla p_2 \tag{51}$$

or

$$\bar{J} \times \bar{B} = \nabla p_2 \ . \tag{52}$$

The macroscopic pressure balance is verified precisely. The striking feature is that this macroscopic relation, which is contained in the guiding-center fluid theory, is obtained microscopically by combining two current components, neither of which has any recognizable connection with the guiding-center fluid derivation in Section 5.

Returning to the specialization of this problem with x-dependence only but with a uniform electric field in the y-direction (cf. Section 4), we easily compute the energy interchange between particles and field as

$$(53) \qquad \begin{cases} \bar{E}\cdot\bar{J}_m = B\ \bar{U}\cdot\nabla(p_2/B) \\ \bar{E}\cdot\bar{J}_d = p_2\bar{U}\cdot\nabla(\log B) \\ \bar{E}\cdot\bar{J} = \bar{U}\cdot\nabla p_2\ . \end{cases}$$

This is quite non-informative, except possibly for the last line which agrees with the corresponding guiding-center fluid result.

For the next example, we take the field derived from a straight line current as in Section 4, but we superpose a uniform Maxwellian plasma filling all space. There is no current flow in the plasma, and the exact orbits must be compatible with the given vacuum magnetic field. But the guiding-center orbits are quite complex. Corresponding to the first order axial drift (17), there is the finite axial current

$$(54) \qquad J_d = ne\langle v_d\rangle = \langle V^2 + \tfrac{1}{2}v_\perp^2\rangle\ (\rho/rB).$$

But there is also a polarization current

$$(55) \qquad \bar{J}_m = \text{curl } \bar{M} = p_2 \bar{B} \times \nabla(1/B^2).$$

The sum, $\bar{J}_d + \bar{J}_m$, must, of course, cancel. But we are presented with a problem in which a macroscopically trivial result is masked by rather complex microscopic considerations.

For the next example, we take a uniform magnetic field in the z-direction together with a uniform but time-varying electric field in the y-direction. The lowest order guiding-center motion is simply a motion in the x-direction, $\bar{U}(t) = \bar{E}(t) \times \bar{B}/B^2$. To next order we have a y-component; from equation (11),

$$(56) \qquad \bar{U}_1 = \frac{1}{B^2} [\bar{E} \times \bar{B} + \delta \frac{d\bar{E}}{dt}].$$

If $\bar{E}(t)$ is oscillatory, the path of a guiding center is an ellipse with small eccentricity (viz., the ratio of the frequency of oscillation of $\bar{E}$ compared to the Larmor frequency), Fig. 1a. If E changes by the definite amount ΔE, the guiding-center changes its speed in the x-direction and is displaced in the y-direction by an amount $\delta(\Delta E)/B^2$, Fig. 1b. Assuming a

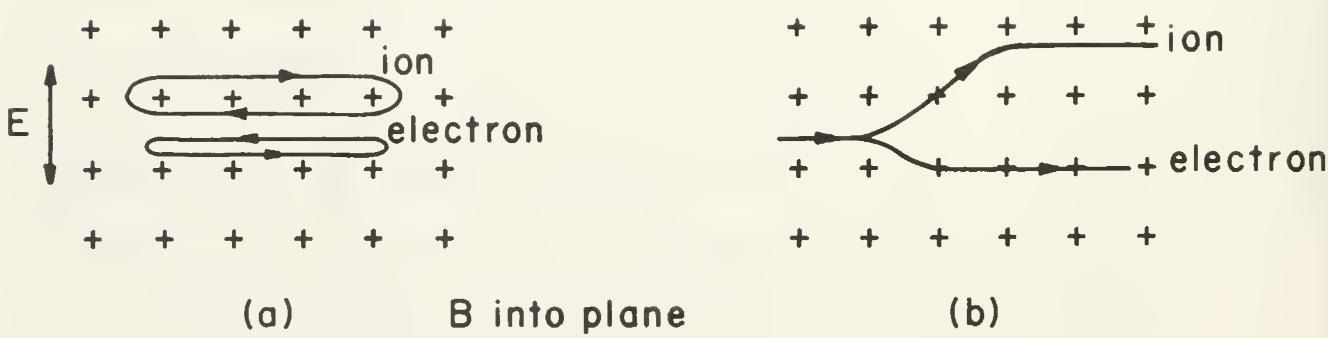

Figure I: ION AND ELECTRON ORBITS

neutral plasma, the current flow from (56) is

$$\bar{J} = \frac{\rho}{B^2}\frac{d\bar{E}}{dt} . \tag{57}$$

If we wish, we can interpret this as a polarization current. In the absence of $\bar{E}$, a given ion and electron attached to the same magnetic line can be considered to form a molecule; with a specified value of $\bar{E}$, they are displaced by a definite amount. From $\bar{J}_p = d\bar{D}/dt - \kappa_o d\bar{E}/dt$ and $\bar{D} = K\,\kappa_o\bar{E}$, we compute an equivalent dielectric constant

$$\left\{\begin{aligned} K &= 1 + \rho/\kappa_o B^2 = 1 + c^2/A^2 \\ A^2 &= B^2/\mu_o\rho \ ; \end{aligned}\right. \tag{58}$$

A is the Alfvèn speed.

Now we turn to a macroscopic guiding-center fluid description of the problem. To be precise, let us consider a perfectly conducting homogeneous fluid placed between two parallel (or concentric cylinder) condenser plates. Any transverse electric field $\bar{E}(t)$ in the fluid must be accompanied by a macroscopic fluid velocity $\bar{u} = (\bar{E}\times\bar{B})/B^2$. By symmetry, there are no pressure gradients; thus the fluid acceleration is balanced by the Lorentz force,

$$\rho\,\frac{du}{dt} = \bar{J}\times\bar{B}. \tag{59}$$

From

(60) $$\frac{d\bar{u}}{dt} = (\frac{d\bar{E}}{dt} \times B)/B^2 ,$$

we obtain exactly the same result as before, (57).

Both analyses are so simple that there is nothing to be gained from one over the other. But is is worth remarking that, while the particle picture describes a drift velocity to two different orders in an expansion parameter, the fluid version separates these into a single tangential velocity coupled with a perpendicular current flow.

The interpretation of a plasma as a dielectric must be used with caution. One can either adopt a super-macroscopic view of the plasma as a black box with a dielectric constant, or peek inside the box and use the equations of motion. The two are equivalent in certain special cases of great symmetry. But the use of the dielectric concept in nonsteady and inhomogeneous situations as a replacement for one drift term among many[20] is decidedly improper for even a heuristic argument. For example, from the value of the dielectric constant, one might deduce that the signal speed has the value A (for $c^2 >> A^2$); but the correct transverse signal speed is $(A^2 + 2RT_2)^{1/2}$. In other words, a dielectric constant which is evaluated in a simple situation in which there are no pressure gradients cannot be correctly used under more general circumstances.

Next, we return to the magnetic field surrounding a line

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20. For example, see footnote Ref. 5.

current but this time study a more realistic problem in which the plasma is of finite extent and is contained within a certain toroidal flux tube. We now present the classical orbit argument which is given to show that such a configuration (viz., a toroidal Stellerator) cannot be used for containment. To simplify matters, we assume that the plasma is tenuous and hardly perturbs the vacuum magnetic field. The field gradient and curvature set up an axial drift, v_d, as before. But the plasma is of finite extent, so a separation of charge and subsequent axial electric field is produced, after which the plasma as a whole moves outward radially with some velocity $\overline{U} = \overline{E} \times \overline{B}/B^2$. Since only guiding-center motion has been referred to, and no mention has been made of heat flow along the magnetic lines, there must be an equivalent macroscopic argument, even though this is not a geometry in which the macroscopic and microscopic theories are identical. Indeed, there is an elementary argument, following from either the magnetohydrodynamic or guiding-center fluid equations, which shows that there exists no equilibrium configuration with the given symmetry unless p (or $p_1 + p_2$) is constant on a cylinder of given radius. In particular, there is no equilibrium which is finite in extent. If one postulates the existence of a macroscopic plasma of finite extent as an initial condition, an outward acceleration can be easily computed by balancing the excess of the external over the internal magnetic pressure against the total mass of the plasma.

The transient problem, looked at microscopically, is more subtle. One must first postulate an undetermined value for the radial acceleration. This acceleration modifies the drifts according to (11). One either computes that value of the acceleration which produces zero mean drift (quasi-neutral approximation) or computes the revised drift current and from it a value for $d\overline{E}/dt$, thus $d\overline{U}/dt$, which latter should be equal to the assumed acceleration.

There is one point at which the microscopic and macroscopic analyses may differ. The electric field produced by the charge separation is essentially a displacement current effect. The macroscopic procedure mimics the quasi-neutral microscopic analysis. There is a reversal of "cause" and "effect". In the first argument, an electric field is created which is then said to induce a motion of the plasma. In the second argument, an unbalanced force results in motion of the plasma with a concomitant electric field (and charge separation which can be computed afterwards). The second argument ignores a rapid transient on the order of the plasma frequency. This is rarely of quantitative importance. Its significance lies only in masking the essential equivalence of a macroscopic investigation of the existence of an equilibrium with a qualitative microscopic description of a transient phenomenon. In all but the simplest geometries, the microscopic argument becomes hopelessly involved, but the investigation of the equilibria is feasible.

Such dual arguments are common, e.g., in some stability

analyses which are deeply involved with the behavior of electric fields but which are substantially equivalent to macroscopic arguments in which the electric field does not even make an appearance.

9. Critique of the Guiding-Center Theory in Static Equilibrium

The requirements of a theory of equilibrium or steady flow are frequently more stringent than those of a general dynamic theory since the solutions are expected to be valid over a relatively long time. For example, a dynamic theory of the pinch might have to be accurate for a few microseconds, but for containment of a plasma, one would hope to find an equilibrium theory that is valid for seconds.

If one wishes to be very cautious in assessing the reliability of a guiding-center theory solution, then, from the fact that the solution may be in error by $O(\delta)$, one can only conclude a containment time which is $O(1/\delta)$ before the solution breaks down. One might be very optimistic, on the other hand, and hope that there exists a neighboring exact solution which the zero-order guiding-center solution approximates. The existence of such an exact self-consistent solution implies infinitely long containment for the particles following their exact orbits (ignoring collisions, of course). Such exact solutions have only been obtained in very special cases, and, from ergodic considerations, one should be doubtful of their existence in any generality. An intermediate possibility

is the existence of equilibria to some higher order than $o(\delta)$ in the guiding-center theory. This is greatly to be desired, since a containment time of order $O(1/\delta)$ is much too short to be practically useful. This is not to say that we cannot hope for much better containment of plasmas, but only to point out that present theory is unable to predict more.

Some further progress can be made by splitting the problem into two parts: first, the analysis of orbits in a given field, and second, the requirement that the field be self-consistent with the orbits. The first problem is much simpler and can shed some light on the second. For example, the existence of an exact equilibrium requires, as a prerequisite, the existence of fields which can contain orbits indefinitely.

From the lowest order theory, which ignores drifts $O(\delta)$, one cannot preclude the possibility of particles moving to the wall in a time $O(1/\delta)$. If the drifts are tangential to some surface which avoids the walls, we can do better. For example, in an axially-symmetric field, the first order drifts are azimuthal and cannot lead out of the system. As a matter of fact, the orbits are absolutely contained in the radial direction and can only emerge axially. Since the adiabatic invariant (magnetic moment to lowest order) is constant to an arbitrarily high order in δ[21] (provided that the field is sufficiently smooth), the individual particle containment is valid to

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21. a) M. D. Kruskal, footnote Ref. 3b; also

b) C. S. Gardner, Phys. Rev. 115, 791 (1959).

arbitrarily high order in $1/\delta$.

Moreover, even in a somewhat asymmetric mirror field, it can be shown that guiding centers are constrained to lie on certain surfaces (approximately flux tubes) to arbitrarily high order in δ. This follows from the existence of a so-called second adiabatic invariant.[22] Thus, although self-consistent equilibria have been only shown to exist to lowest order,[23] the orbit theory at least does not rule out the possibility of much better results.

One might ask, if it has been proved that particles are contained for a long time in quite general fields, why worry about self-consistency? The reason is that the proof of containment depends on time-independence or, at most, slow time variation of the field. Lack of self-consistency could vitiate this assumption.

For a toroidal configuration (e.g., a Stellerator) similar results can be proved, viz., the existence of guiding-center surfaces to arbitrarily high order.[24] We are content here to quote the weaker result that, although the drifts are not

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22. The idea of such an invariant is attributed to M. Rosenbluth; its existence was proved by C. S. Gardner, footnote Ref. 21b, also by T. G. Northrup and E. Teller, Phys. Rev. 117, 215 (1960).

23. For example see A. Oppenheim, "Equilibrium Configuration of a Plasma in a Guiding Center Limit", Institute of Mathematical Sciences Report NYO-9353, September 1960.

24. Unpublished.

tangential to a flux tube, the average value of the normal drift component on a pressure surface is zero (see Appendix). This is sufficient to guarantee a particle containment which is large compared to $1/\delta$.

The error made in self-consistency, viz., $O(\delta)$, can be improved if we restrict ourselves to low pressure plasmas. A vacuum field is, of course, exactly self-consistent. In a low pressure plasma in which $J = O(\delta)$, the error in self-consistency of the lowest order solution becomes $O(\delta^2)$. If one wished to make use of more accurate orbit results, say $O(\delta^r)$, in a consistent way, one might sacrifice the plasma pressure to be $O(\delta^{r-1})$; in other words, the plasma pressure is inversely proportional to the required containment time, using the best available theory.

The self-consistency problem becomes very difficult when carried to higher order. One reason is that the surfaces on which the particles are constrained no longer fill the space simply; there is a different, mutually intersecting family of surfaces for each value of the energy and magnetic moment.

Our general philosophy is not to cast doubt on the possibility of containing plasmas, but to delimit the predictions which can be made with current theory. These limitations are independent of any stability considerations. A self-consistent equilibrium may be valid to very high order and be unstable; or it may not exist to second order even though the first order solution is found to be stable.

APPENDIX: Average Drifts on a Toroidal Surface

We suppose that there is given a toroidal solution of $\nabla p = \text{curl}\ \bar{B} \times \bar{B}/\mu_o$ [25]; i.e., a solution with nested toroidal pressure surfaces which are also flux tubes.[26] The drift velocity in the absence of an electric field is

$$\bar{v}_d = \frac{\delta}{B^2} \bar{B} \times [(\bar{V} \cdot \nabla)\bar{V} + \mu \nabla B]. \tag{A.1}$$

Using the identity

$$\nabla p \cdot \bar{B} \times (\bar{B} \cdot \nabla)\bar{B} = \nabla p \cdot \bar{B} \times \nabla(\tfrac{1}{2} B^2), \tag{A.2}$$

we can write the rate of change of p following the particle as

$$\frac{dp}{dt} = \bar{v}_d \cdot \nabla p = \frac{\delta}{B^3} (V^2 + \tfrac{1}{2} v_\perp^2)(\nabla p \cdot \bar{B} \times \nabla B) \tag{A.3}$$

$$= \frac{\delta}{B^3} (W^2 - \mu B)(\nabla p \times \bar{B} \cdot \nabla B)$$

where

$$W^2 = V^2 + v_\perp^2 \tag{A.4}$$

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25. It is easily shown that an appropriate guiding-center distribution function can be found which corresponds to any given macroscopic, scalar-pressure, toroidal equilibrium.

26. a) H. Grad and H. Rubin, Proceedings of the Second Internat'l Conf. on the Peaceful Uses of Atomic Energy, Geneva, 1958, Vol. 31, p. 190; also,

    b) M. D. Kruskal and R. M. Kulsrud, ibid., p. 213.

is a constant of the motion, as is μ. It is convenient to introduce new variables (p, ω, ϕ) for $(\bar{x})$ where

$$(A.5) \qquad \bar{B} = \nabla p \times \nabla \omega = \nabla \phi - n(n \cdot \nabla \phi);$$ [27]

n is the unit normal, $\nabla p / |\nabla p|$, and the element of flux is given by $\bar{B} \cdot d\bar{S} = dp d\omega$. Part of the transformation of base vectors is

$$(A.6) \qquad B^2 \frac{\partial \bar{x}}{\partial \omega} = \nabla \phi \times \nabla p = \bar{B} \times \nabla p$$

from which we obtain

$$(A.7) \qquad \frac{dp}{dt} = -\delta (W^2 - \mu B) \frac{\partial}{\partial \omega} \log B.$$

The mean change in p after a long interval of time

$$(A.8) \qquad [p(t) - p(0)]/t = \langle dp/dt \rangle = \frac{1}{t} \int_0^t \left(\frac{dp}{dt}\right) dt$$

can be replaced by an ergodic average with respect to the invariant measure which, in three dimensions, is

$$(A.9) \qquad dm = (\bar{B} \cdot d\bar{S}) dt$$

where dt is the time interval along an orbit. From $dt = ds/V = d\bar{\phi}/BV$, we conclude that

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27. See footnote Ref. 26a.

(A.10) $$\left\langle \frac{dp}{dt} \right\rangle = \int \frac{\partial Q}{\partial \omega}\, dp\, d\omega\, d\phi$$

where

(A.11) $$Q(B) = -\delta \int_{B_o}^{B} \frac{W^2 - \mu B}{(W^2 - 2\mu B)^{1/2}} \frac{dB}{B^2}\,.$$

The integration in (A.10) extends over the volume between two p-surfaces. For the correct ergodic average on a p-surface, we need only integrate with respect to ω and ϕ. From the form of the integrand (despite the fact that ω and ϕ are not single-valued on a torus) we easily conclude that $\langle dp/dt \rangle = 0$.

The above argument indicates that the eventual rate at which a particle moves away from a pressure surface is small compared to δ. One would expect to obtain a stronger and more precise result, viz., that the drift is $O(\delta^2)$, by use of an appropriate adiabatic invariant. This type of result is much more subtle, however, and requires more refined estimates about higher order guiding-center motion; (e.g., in the non-toroidal case, see the references of footnote 22). A toroidal adiabatic invariant has been introduced by Kulsrud[28], but it is inadequate for such a purpose since it is shown to be invariant for only the lowest order guiding-center motion. From this fact it is easily shown that this invariant supplies no information at all about containment in

28. R. M. Kulsrud, Phys. Fluids 4, 302 (1961).

a strictly time-independent field and, in a time-dependent problem, only if the time variation, while slow, is fast compared to the spatial variation (say $O(\delta^{1/2})$). The latter result can be shown to follow from the existence of an exact constant of the lowest order motion which is closely related to Kulsrud's adiabatic invariant. We see this as follows.

A concentric nest of toroidal flux surfaces is postulated. According to the lowest order (flux preserving) motion, this topological property is preserved with time even if the motion is not slow. Further, it is easily seen that the property of a particular surface being covered ergodically by its magnetic lines is an invariant property of the surface.

From the conventional analysis, one is led to consider an integral of the form (cf. (10))

$$\int V\, ds = \int p d\sigma$$

(hereafter, p will denote momentum, not pressure), where the integral is taken along a magnetic line, but the limits are rather imprecise on a torus. We can write

$$\int p\, d\sigma = \int p(d\sigma/dt)dt = \int \zeta^2 p^2 dt$$

and, tentatively, replace the time integral by a volume integral with respect to the invariant measure. We write

$$\bar{B} = \nabla\alpha \times \nabla\gamma$$

where α is constant on a flux tube (the pressure is not

constant on a flux tube in a general equilibrium), and observe that

$$dm = d\alpha d\gamma d\sigma/\zeta^2 p,$$

which leads us to the definition

$$I = \int p\, d\alpha\, d\gamma\, d\sigma = \int VB\, dx.$$

The variables γ and σ are integrated over the complete surface of a torus, and α is integrated between two fixed but arbitrary values. Or, the α integration can be omitted. For an arbitrary time variation of the electromagnetic field, we assume that the initial momentum p is given as an arbitrary function of (α, γ, σ), and each of these initial values is followed in time as a solution of the guiding-center equations. Thus I becomes a function of time. Since (α, γ, σ) are Lagrangian (fixed) coordinates with respect to the motion, we may differentiate under the integral sign, obtaining

$$\frac{dI}{dt} = \int \frac{dp}{dt}\, d\alpha\, d\gamma\, d\sigma$$

$$= -\int \frac{\partial H}{\partial \sigma}\, d\alpha\, d\gamma\, d\sigma$$

which clearly vanishes. Thus I is an exact constant of the lowest order motion.